EXERCISES: CENTRAL BANKS, MONETARY POLICY AND RISK

HARJOAT S. BHAMRA

(Additional question to be added)

Question 1. Time is continuous, $t \in [0, \infty)$. There is a representative household with preferences defined over consumption rate and work flow. Her date-t expected utility is given by

$$E_t \int_t^{\infty} e^{-\delta(u-t)} \left(\ln C_u - \frac{N_u^{1+\varphi}}{1+\varphi} \right) du, \tag{1}$$

where output flow is given by

$$Y_t = A_t N_t, \tag{2}$$

where A_t is the level of technological progress, given by

$$\frac{dA_t}{A_t} = \mu dt + \sigma dZ_t, \tag{3}$$

where Z is a standard Brownian motion under the physical measure $\mathbb{P}$

In this economy, there are no frictions and we shall assume that the representative household can invest its financial wealth in two securities. The first is a nominal bond, which pays of 1 USD at an instant from now. The second is a claim on output flow.

The nominal rate of return on the nominal bond is $i_t = r_t + \pi_t$, where r_t is the real risk-free rate of return and π_t is the rate of inflation. Consequently, the real rate of return on the nominal bond is r_t.

Denoting the real date-t price of the claim to output flow, the real cum-dividend return on this claim over an instant dt is

$$\frac{dS_t + A_t N_t dt}{S_t}. \tag{4}$$

(1) Derive the household's dynamic budget constraint.
(2) Starting from the HJB equation for the household, show that the household's value function is given by

$$V_t = \frac{1}{\delta} \left(a_t - \frac{1}{1+\varphi} + \frac{\mu - \frac{1}{2}\sigma^2}{\delta} \right). \tag{5}$$

Question 2. Time is continuous, $t \in [0, \infty)$. There is a representative agent with preferences defined over consumption rate and work flow. Her date-t expected utility is given by

$$E_t \int_t^{\infty} e^{-\delta(u-t)} \left(\ln C_u - \frac{N_u^{1+\varphi}}{1+\varphi} \right) du, \tag{6}$$

where

$$C_t = \left(\int_{i \in [0,1]} C_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{1}{1-\frac{1}{\epsilon}}}, \tag{7}$$

and $C_t(i)$ is the household's rate of consumption for good i. Observe that ϵ is the elasticity of substitution between any two goods.

Differentiated goods are produced by a continuum of firms, $i \in [0,1]$, where firm i's date-t output flow is $Y_t(i)$, where

$$Y_t(i) = A_t N_t(i), \tag{8}$$

and $N_t(i)$ is firm i's labor input flow and A_t is the level of technological progress, which is common across firms, and is given by

$$\frac{dA_t}{A_t} = \mu dt + \sigma dZ_t, \tag{9}$$

where Z is a standard Brownian motion under the physical measure $\mathbb{P}$.

In this economy, there is only one friction – firms have monopoly power, because they produce differentiated goods. Recall that monopolistic competition is a type of imperfect competition where many producers sell products that are differentiated from one another (e.g. by branding or quality) and hence are not perfect substitutes. In monopolistic competition, a firm takes the prices charged by its rivals as given and ignores the impact of its own prices on the prices of other firms. The particular model of monopolistic competition used here is based on Dixit and Stiglitz (1977). When used in this context, the CES aggregator in (7) is commonly referred to as the the Dixit-Stiglitz aggregator.

In the limit, as $\epsilon \to \infty$, goods are perfect substitutes for each other and so firms have no monopoly power. This is the perfectly competitive limit. In this case, (7) reduces to

$$C_t = \int_{i \in [0,1]} C_t(i) di. \tag{10}$$

We shall assume that the representative household can invest its financial wealth in two securities. The first is a nominal bond, which pays of 1 USD at an instant from now. The second is a claim on aggregate real output flow, Y_t, given by

$$Y_t = \int_{i \in [0,1]} Y_t(i) = A_t \int_{i \in [0,1]} N_t(i) = A_t N_t \tag{11}$$

where

$$N_t = \int_{i \in [0,1]} N_t(i). \tag{12}$$

Observe that $D_t(i)$ is the date-t real dividend flow paid by firm i:

$$D_t(i) = \frac{P_t(i)}{P_t} Y_t(i) - \frac{W_t}{P_t} N_t(i). \tag{13}$$

and $\frac{W_t}{P_t}$ is the real wage rate. We know that $N_t(i) = \frac{Y_t(i)}{A_t}$, and so

$$D_t(i) = \left(\frac{P_t(i)}{P_t} - \frac{W_t}{A_t P_t}\right) Y_t(i). \tag{14}$$

Firm i chooses dividends in order maximize their expected present value, given at date-t by

$$S_t(i) = E_t \int_t^{\infty} \frac{\Lambda_u}{\Lambda_t} D_u(i) du, \tag{15}$$

where Λ is the equilibrium real SDF process.

The real rate of return on the claim to aggregate output flow over an instant dt is given by

$$\frac{dJ_t + Y_t dt}{J_t}, \tag{16}$$

where J_t is the date-t real price of the claim to aggregate output flow. Observe that

$$Y_t = D_t + \frac{W_t}{P_t} N_t, \tag{17}$$

where the represenative household takes date-t aggregate dividend flow, D_t, defined by

$$D_t = \int_{i \in [0,1]} D_t(i), \tag{18}$$

as given.

The household owns the claim to aggregate output and takes the dividend component of output flow as given, because this is chosen by firms. The household also takes the real wage rate $\frac{W_t}{P_t}$ as given. However, the household chooses her work flow N_t.

To solve for equilibrium prices and quantities, we shall consider the optimal stochastic control problems of firms and the representative household separately, and then impose market clearing. We are solving for the decentralized equilibrium. In the previous question, where there was only one firm, we considered only the household's problem – we solved for equilibrium by considering the household as a social planner who made decisions for her own benefit. As part of this question, we shall see that in the perfectly competitive limit, equilibrium quantities and prices are the same as in the previous question. However, with imperfect competition there is a difference – aggregate employment is distorted.

(1) We start by considering the household's problem. Derive the household's dynamic intertemporal budget constraint.
(2) Write down the HJB for the representative household. Hence show that the represenative household's value function is given by

$$V_t = \frac{1}{\delta}\left[a_t + \ln N_t + \frac{N_t^{1+\varphi}}{1+\varphi} - \frac{1}{\delta}\left(\mu - \frac{1}{2}\sigma^2\right)\right], \tag{19}$$

where

$$N_t^{1+\varphi} = \frac{W_t}{P_t A_t}. \tag{20}$$

(3) By considering the optimal stochastic control problem of firm i, show that the firm's optimal choice of price is given by

$$P_t(i) = \frac{1}{1 - \frac{1}{\epsilon}} \frac{W_t}{A_t}. \tag{21}$$

Explain the economics underlying firms' use of their monopoly power. Show how monopoly power affects aggregate employment, aggregate consumption demand, aggregate output and the real wage rate.