Lecture 3: Liquidity Traps

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Model Outline

Benhabib, Schmitt-Grohe & Uribe (2002)

- No risk
- Constant aggregate endowment (no growth either)
- Continuum of identical households – derive utility flows from consumption flows and real money balances
- Government
- Fiscal and monetary policy
- Assets: money, nominal bonds and claim to aggregate endowment

Aggregate Endowment

- If we switch off risk in the standard Lucas endowment economy (where output is log-normal), we still have exogenous growth

$$Y_t = Y_0 e^{\mu_Y t} \tag{1}$$

- We shall also ignore growth: $\mu_Y = 0$

Assets I

- Nominal bonds in zero-net supply, with date-t nominal price, $B_t^{\$}$

$$\frac{dB_t^{\$}}{B_t^{\$}} = i_t dt \tag{2}$$

- Real price of the nominal bond, $B_t = \frac{B_t^{\$}}{P_t}$, where evolution of price index is determined by inflation, π

$$\frac{dP_t}{P_t} = \pi_t dt \tag{3}$$

- It follows that

$$\frac{dB_t}{B_t} = r_t dt, \tag{4}$$

where

$$i_t = r_t + \pi_t \tag{5}$$

Assets II

- The date-t nominal price of the claim to the aggregate endowment is $S_t^\$$. By no arbitrage the nominal return must equal the nominal interest rate

$$\frac{dS_t^\$ + P_t Y_t dt}{S_t^\$} = i_t dt \tag{6}$$

- Real return

$$\frac{dS_t + Y_t dt}{S_t} = r_t dt \tag{7}$$

- Price asset using discount factor (remember $Y_t = Y$)

$$S_t = Y \int_t^\infty e^{-\int_t^u r_s ds} du = Y p_t, \tag{8}$$

where

$$p_t = e^{-\int_t^u r_s ds} du, \tag{9}$$

Assets III

and so

$$dS_t = Ydp_t. \tag{10}$$

Hence

$$S_t^{\$} = P_t Y p_t \tag{11}$$

$$\frac{dS_t^{\$}}{S_t^{\$}} = \frac{dP_t}{P_t} + \frac{dp_t}{p_t} = \pi_t dt + \frac{dp_t}{p_t} \tag{12}$$

Households I

- Continuum of identical households – can work with represenative agent

$$\int_t^{\infty} e^{-\delta(u-t)} u\left(C_u, \frac{M_u}{P_u}\right) du \tag{13}$$

  - $u(\cdot,\cdot)$ increasing in both arguments and concave
  - $u_{C,M/P} > 0$ making consumption and real money balances Edgeworth complements
  -

$$\forall Y > 0, \lim_{M/P \to \infty} \frac{u_{M/P}(Y, M/P)}{u_C(Y, M/P)} \leq R(\pi_L) \tag{14}$$

- Rep household invests in nominal debt (in zero net supply) and claim on aggregate endowment
- Date-t nominal wealth

$$W_t^{\$} = N_{B,t} B_t^{\$} + N_{S,t} S_t^{\$} + M_t \tag{15}$$

- Rep household can consume and also pays taxes – derive dynamic intertemporal budget constraint

Households II

- in real terms, dynamic intertemporal budget constraint is

$$dW_t = r_t(W_t - N_{S,t}S_t)dt - i_t\frac{M_t}{P_t}dt + N_{S,t}(dS_t + Y_t)dt - C_t dt - T_t dt \qquad (16)$$

which is equivalent to

$$\dot{W}_t = r_t(W_t - N_{S,t}S_t) - i_t\frac{M_t}{P_t} + N_{S,t}(dS_t + Y_t) - C_t - T_t \qquad (17)$$

Household Budget Constraint I

$$W_t^{\$} = N_{B,t}B_t^{\$} + N_{S,t}S_t^{\$} + M_t \tag{18}$$

$$W_{t+dt}^{\$} = N_{B,t+dt}B_{t+dt}^{\$} + N_{S,t+dt}(S_{t+dt}^{\$} + P_tY_tdt) + M_{t+dt} - P_tC_tdt - P_tT_tdt \tag{19}$$

$$\begin{aligned} W_{t+dt}^{\$} &= N_{B,t}B_t^{\$} + B_t^{\$}dN_{B,t} + N_{B,t}dB_t^{\$} && (20)\\ &+ N_{S,t}S_t^{\$} + S_t^{\$}dN_{S,t} + N_{S,t}(dS_t^{\$} + P_tY_tdt) + M_t + dM_t && (21)\\ &- P_tC_tdt - P_tT_tdt && (22)\\ & && (23)\end{aligned}$$

We know

$$dS_t^{\$} = S_t^{\$}\left(\pi_t dt + \frac{dp_t}{p_t}\right) \tag{24}$$

Household Budget Constraint II

and

$$dB_t^{\$} = B_t^{\$} i_t dt \tag{25}$$

and so

$$dW_t^{\$} = N_{B,t} B_t^{\$} i_t dt + N_{S,t} \left(S_t^{\$} \pi_t dt + S_t^{\$} \frac{dp_t}{p_t} + P_t Y_t dt \right) - P_t C_t dt - P_t T_t dt \tag{26}$$

$$+ dM_t + B_t^{\$} dN_{B,t} + S_t^{\$} dN_{S,t} \tag{27}$$

Households cannot create assets only exchange them for other assets, so $B_t^{\$} dN_{B,t} + S_t^{\$} dN_{S,t} + dM_t = 0$

$$dW_t^{\$} = N_{B,t} B_t^{\$} i_t dt + N_{S,t} \left(S_t^{\$} \pi_t dt + S_t^{\$} \frac{dp_t}{p_t} + P_t Y_t dt \right) - P_t C_t dt - P_t T_t dt \tag{28}$$

Household Budget Constraint III

Now express budget constraint in real terms

$$W_t = \frac{W_t^{\$}}{P_t} \tag{29}$$

$$dW_t = \frac{dW_t^{\$}}{P_t} - \pi_t W_t dt \tag{30}$$

Household Budget Constraint IV

Therefore

$$dW_t = N_{B,t}B_t i_t dt + N_{S,t}\left(S_t\pi_t dt + S_t\frac{dp_t}{p_t} + Y_t dt\right) - C_t dt - T_t dt - \pi_t W_t dt \tag{31}$$

$$dW_t = N_{B,t}B_t i_t dt + N_{S,t}\left(S_t\pi_t dt + S_t\frac{dp_t}{p_t} + Y_t dt\right) - C_t dt - T_t dt \tag{32}$$

$$- \pi_t\left(N_{B,t}B_t + N_{S,t}S_t + \frac{M_t}{P_t}\right) dt \tag{33}$$

$$= N_{B,t}B_t r_t dt - \pi_t\frac{M_t}{P_t}dt + N_{S,t}\left(S_t\frac{dp_t}{p_t} + Y_t dt\right) - C_t dt - T_t dt \tag{34}$$

$$= \left(N_{B,t}B_t + \frac{M_t}{P_t}\right) r_t dt - i_t\frac{M_t}{P_t}dt + N_{S,t}\left(S_t\frac{dp_t}{p_t} + Y_t dt\right) - C_t dt - T_t dt \tag{35}$$

$$= \left(W_t - N_{S,t}S_t\right) r_t dt - i_t\frac{M_t}{P_t}dt + N_{S,t}\left(S_t\frac{dp_t}{p_t} + Y_t dt\right) - C_t dt - T_t dt \tag{36}$$

Matching up with Benhabib, Schmitt-Grohe & Uribe (2002)

Define

$$A_t = N_{B,t} B_t + \frac{M_t}{P_t} \tag{37}$$

and so (ex dividend flows)

$$\begin{aligned} dW_t &= dA_t + N_{S,t} dS_t && (38) \\ &= dA_t + N_{S,t} Y dp_t && (39) \\ &= dA_t + N_{S,t} S_t \frac{dp_t}{p_t} && (40) \end{aligned}$$

Therefore

$$dA_t = A_t r_t dt - i_t \frac{M_t}{P_t} dt + N_{S,t} Y_t dt - C_t dt - T_t dt \tag{41}$$

[This is equation (2) in Benhabib, Schmitt-Grohe & Uribe (2002)]

Government's Budget Constraint I

- Government finances deficits by printing money, M and issuing nominal bonds with price, $B^{\$}$. Public consumption is assumed to be zero and the government levies fixed real taxes T per unit time.

$$W_t^{G,\$} = N_{B,t}^{G} B_t^{\$} - M_t \tag{42}$$

Government's Budget Constraint II

$$W_{t+dt}^{G,\$} = N_{B,t+dt}^{G} B_{t+dt}^{\$} - M_{t+dt} + P_t T_t dt \quad (43)$$

$$W_{t+dt}^{G,\$} = (N_{B,t}^{G} + dN_{B,t}^{G})(B_t^{\$} + dB_t^{\$}) - M_t - dM_t + P_t T_t dt \quad (44)$$

$$W_{t+dt}^{G,\$} = N_{B,t}^{G} B_t^{\$} + N_{B,t}^{G} dB_t^{\$} + dN_{B,t}^{G} B_t^{\$} - M_t - dM_t + P_t T_t dt \quad (45)$$

$$dW_t^{G,\$} = N_{B,t}^{G} dB_t^{\$} + dN_{B,t}^{G} B_t^{\$} - dM_t + P_t T_t dt \quad (46)$$

$$dW_t^{G,\$} = N_{B,t}^{G} dB_t^{\$} + dN_{B,t}^{G} B_t^{\$} - dM_t + P_t T_t dt \quad (47)$$

$$dW_t^{G,\$} = N_{B,t}^{G} B_t^{\$} i_t dt + dN_{B,t}^{G} B_t^{\$} - dM_t + P_t T_t dt \quad (48)$$

$$dW_t^{G,\$} = (W_t^{G,\$} + M_t) i_t dt + dN_{B,t}^{G} B_t^{\$} - dM_t + P_t T_t dt \quad (49)$$

The central bank is an arm of the government and can print money to purchase nominal bonds, so $dN_{B,t}^{G} B_t^{\$} - dM_t = 0$

$$dW_t^{G,\$} = W_t^{G,\$} i_t dt + M_t i_t dt + P_t T_t dt \quad (50)$$

Government's Budget Constraint III

In real terms

$$dW_t^G = W_t^G r_t dt + \frac{M_t}{P_t} i_t dt + T_t dt \quad (51)$$

In equilibrium

$$A_t + W_t^G = 0 \quad (52)$$

and so

$$dA_t = A_t r_t dt - \frac{M_t}{P_t} i_t dt - T_t dt \quad (53)$$

[This is equation (9) in Benhabib, Schmitt-Grohe & Uribe (2002)]
The central bank imposes the following interest rate rule

$$i_t = R(\pi_t), \quad (54)$$

where $\forall \pi$, $R(\pi) \geq 0$ and

$$\exists \pi^* > -\delta : R(\pi^*) = \delta + \pi^*, R'(\pi^*) > 1 \quad (55)$$

Government's Budget Constraint IV

The nominal interest rate can never go negative and monetary policy is active (see Taylor (1993)) in the sense that around the inflation target, π^*, because the central bank responds to increases (decreases) in inflation with a more than one-for-one increase (decrease) in the nominal interest rate.

Household's Deterministic Optimal Control Problem I

- $t \in \mathcal{T} = [0, \infty)$
- We have a 1-d state, A, which evolves over time according to the following law of motion

$$dA_t = (R(\pi_t) - \pi_t)dt - R(\pi_t)\frac{M_t}{P_t} + Y - C_t - T \tag{56}$$

- The starting value of the state is given by $A(0) = A_0$. The future values of the state will depend on the control variables C_t and M_t/P_t.
- The household chooses the path of the controls, $(C_t, M_t/P_t)_{t \in \mathcal{T}}$. Her objective is to maximize the discounted value of her utility flows. At time-t, the utility flow function is given by

$$u(C_t, M_t/P_t) \tag{57}$$

- With a constant time discount rate δ, the household's objective is given by

$$J(A_0) = \sup_{(C_t, M_t/P_t)_{t \in \mathcal{T}}} \int_0^\infty e^{-\delta t} u(C_t, M_t/P_t)dt \tag{58}$$

- Date-t objective function

$$J_t = J(A_t) = \sup_{(C_u, M_u/P_u)_{u \geq t}} \int_t^\infty e^{-\delta(u-t)} u(C_u, M_u/P_u)du \tag{59}$$

- What path should the household choose?

Pontryagin's Maximum Principle I

- HJB equation

$$0 = \sup_{C_t, M_t/P_t} u(C_t, M_t/P_t) - \delta J(A_t) + J'(A_t)\left[(R(\pi_t) - \pi_t)dt - R(\pi_t)\frac{M_t}{P_t} + Y - C_t - T\right] \tag{60}$$

$$+ \text{terms involving exogenous state variables, which household does not control} \tag{61}$$

- Maximum Principle

$$\mathcal{H}(A_t, C_t, M_t/P_t, \hat{\Lambda}_t) = u(C_t, M_t/P_t) + \hat{\Lambda}_t\left[(R(\pi_t) - \pi_t)dt - R(\pi_t)\frac{M_t}{P_t} + Y - C_t - T\right] \tag{62}$$

$$\mathcal{H}_C(A_t, C_t, M_t/P_t, \hat{\Lambda}_t) = 0, \text{ given } s_0 \tag{63}$$

$$\mathcal{H}_{M/P}(A_t, C_t, M_t/P_t, \hat{\Lambda}_t) = 0, \text{ given } s_0 \tag{64}$$

$$\mathcal{H}_A(A_t, C_t, M_t/P_t, \hat{\Lambda}_t) + \frac{d\hat{\Lambda}_t}{dt} - \delta\hat{\Lambda}_t = 0 \tag{65}$$

$$\lim_{T\to\infty} e^{-\delta(T-t)}\hat{\Lambda}_T A_T = 0, \tag{66}$$

Pontryagin's Maximum Principle II

- Household Optimality Conditions

$$u_C(C_t, M_t/P_t) = \hat{\Lambda}_t, \text{ given } A_0 \tag{67}$$

$$u_{M/P}(C_t, M_t/P_t) = R(\pi_t)\hat{\Lambda}_t, \text{ given } A_0 \tag{68}$$

$$\frac{d\hat{\Lambda}_t}{dt} + (R(\pi_t) - \pi_t - \delta) = 0 \tag{69}$$

$$\lim_{T\to\infty} e^{-\delta(T-t)}\hat{\Lambda}_T A_T = 0, \tag{70}$$

Market clearing I

- $C_t = Y$
- Eliminate M/P from

$$u_C(C_t, M_t/P_t) = \hat{\Lambda}_t, \text{ given } A_0 \quad (71)$$

$$u_{M/P}(C_t, M_t/P_t) = R(\pi_t)\hat{\Lambda}_t, \text{ given } A_0 \quad (72)$$

$$(73)$$

- Obtain liquidity preference function – links real money balances to output and nominal interest rate

$$\frac{M_t}{P_t} = l(Y, R(\pi_t)), \; l_Y > 0, \; l_R < 0 \quad (74)$$

  - Output flow is constant, so dynamics of real money balance are determined by dynamics of inflation

Market clearing II

- Using FOC for C

$$\hat{\Lambda}_t = L(R(\pi_t)),\ L' < 0 \tag{75}$$

- Sub'ing into ode for $\hat{\Lambda}_t$ gives ode for inflation

$$\underbrace{R'(\pi_t)}_{>0}\frac{d\pi_t}{dt} = \underbrace{\frac{-L(R(\pi_t))}{L'(R(\pi_t))}}_{>0}(R(\pi_t) - \pi_t - \delta) \tag{76}$$

- In above ode, the independent variable is time and there is only one dependent variable, inflation. The household's wealth, excluding the claim on the endowment is given an ode, which which depends in inflation. Hence, the dynamics of inflation impact the dynamics of the household's balance sheet, but the dynamics of the balance sheet do not impact inflation.

$$\frac{dA_t}{dt} = R(\pi_t) - \pi_t - R(\pi_t)\frac{M_t}{P_t} + \overbrace{Y - C_t}^{=0} - T \tag{77}$$

Market clearing III

- Suppose fiscal policy is given by

$$T + R(\pi_t)\frac{M_t}{P_t} = \alpha A_t \tag{78}$$

Consequently

$$\frac{dA_t}{dt} = (R(\pi_t) - \pi_t - \alpha)A_t \tag{79}$$

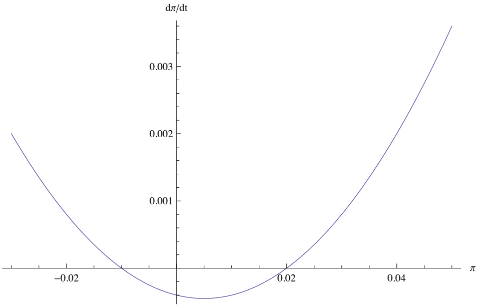
Summary of equilibrium dynamics

$$\frac{dA_t}{dt} = (R(\pi_t) - \pi_t - \alpha)A_t, \text{ given } A_0 \quad (80)$$

$$\underbrace{R'(\pi_t)}_{>0} \frac{d\pi_t}{dt} = \underbrace{\frac{-L(R(\pi_t))}{L'(R(\pi_t))}}_{>0}(R(\pi_t) - \pi_t - \delta) \quad (81)$$

Taylor rule & inflation dynamics I

- We have a Taylor type interest rate rule, where $R(\pi)$ is an increasing function of π. Furthemore, $R''(\pi) > 0$ above π^*, but $R''(\pi) < 0$ below π^*



- Two critical points (steady states), which are the roots of $R(\pi) = \pi + \delta$
  - target inflation, π^*, unstable critical point, because $d\pi/dt > 0$ for $\pi > \pi^*$ and $d\pi/dt < 0$ for $\pi^L < \pi < \pi^*$
  - liquidity trap, π^L, stable critical point, because $d\pi/dt < 0$ for $\pi^L < \pi < \pi^*$ and $d\pi/dt > 0$ for $\pi < \pi^L$

Taylor rule & inflation dynamics II

- Central banks aims to use Taylor use to meet inflation target of π^*, but the zero lower bound constraint implies that $R'(\pi)$ must be decreasing at some point when inflation is below target.
- The zero lower bound constraint in tandem with the Taylor rule creates a second steady state, which is lower than the target and is also an attractor. Therefore, if inflation falls below target, the economy gets pulled into a low inflation and low interest rate equilibrium, commonly referred to as a **liquidity trap**
- If inflation is above target, $d\pi/dt > 0$, so inflation keeps on increasing, creating **hyperinflation**.

Wealth dynamics and fiscal policy I

- Household's wealth dynamics are impacted by inflation dynamics combined with fiscal policy

$$\frac{dA_t}{dt} = (R(\pi_t) - \pi_t - \alpha)A_t, \text{ given } A_0 \tag{82}$$

1. Low tax regime: $\alpha < r$ (fiscal policy parameter is α, which is tax revenues plus interest on real money balances relative to household's real wealth excluding endowment claim)
    - When $\pi > \pi^*$, $R(\pi_t) - \pi_t - \delta > 0$ and so $\frac{dA_t}{dt} > 0$: on a hyperinflationary trajectory, household wealth increases
    - When $\pi^L < \pi < \pi^*$, $R(\pi_t) - \pi_t - \delta < 0$ and so $\frac{dA_t}{dt} > 0$ at first as inflation declines, and then turns negative: during deflation caused by liquidity trap, household wealth eventually declines
    - When $\pi < \pi^L$, $R(\pi_t) - \pi_t - \delta > 0$ and so $\frac{dA_t}{dt} > 0$: as inflation rises, household wealth rises too

Wealth dynamics and fiscal policy II

2. High tax regime: $\alpha > \delta$ (fiscal policy parameter is α, which is tax revenues plus interest on real money balances relative to household's real wealth excluding endowment claim)
   - When $\pi > \pi^*$, $R(\pi_t) - \pi_t - \delta > 0$ and so $\frac{dA_t}{dt} < 0$ at first and turns positive when inflation is sufficiently high: on a hyperinflationary trajectory, household wealth initially decreases, but eventually increases
   - When $\pi^L < \pi < \pi^*$, $R(\pi_t) - \pi_t - \delta < 0$ and so $\frac{dA_t}{dt} < 0$: during deflation caused by liquidity trap, household wealth declines
   - When $\pi < \pi^L$, $R(\pi_t) - \pi_t - \delta > 0$ and so $\frac{dA_t}{dt} > 0$: during a severe deflation, but if inflation rises beyond a given leve below π^L, household wealth will start to decline

What about the stock market? I

- We have not thought about the claim to the endowment. There is no risk and the endowment is a constant, so this claim (the stock market) is at first blush not particularly interesting. We shall actually see that the dynamics of the price-dividend ratio are strongly impacted by the liquidity trap
- Still, we can solve for the value of the stock market. We now know that the price-dividend ratio, p_t, is a function of inflation, i.e. $p_t = p(\pi_t)$.
- No arbitrage implies that

$$\frac{dS_t}{dt} + Y = (R(\pi_t) - \pi_t)S_t, \tag{83}$$

and so

$$p'(\pi_t)\frac{d\pi_t}{dt} + 1 = (R(\pi_t) - \pi_t)p(\pi_t) \tag{84}$$

- Using our ode for π, we obtain an ode for p as a function of inflation

$$p'(\pi_t)\frac{-L(R(\pi_t))}{L'(R(\pi_t))R'(\pi_t)}(R(\pi_t) - \pi_t - \delta) + 1 = (R(\pi_t) - \pi_t)p(\pi_t) \tag{85}$$

- We know the economy has two steady states, $\pi^L < \pi^*$, where $R(\pi_t) - \pi_t - \delta = 0$, and so

What about the stock market? II

- in the liquidity trap

$$p(\pi^L) = \frac{1}{R(\pi^L) - \pi^L} \tag{86}$$

- at the inflation target

$$p(\pi^*) = \frac{1}{R(\pi^*) - \pi^*} \tag{87}$$

- We know that $R(\pi^*) > R(\pi^L)$, so $R(\pi^L) - \pi^L > R(\pi^*) - \pi^*$ if π^L is sufficiently negative.
- Therefore if deflation is sufficiently severe during the liquidity trap, the stock market crashes (relative to normal times when inflation is close to its target), i.e. $p(\pi^L) < p(\pi^*)$.

What can be done while in a liquidity trap?

Werning (2012)

- three equation model (see lecture 1)

$$\frac{dx_t}{dt} = \sigma^{-1}(i_t - \pi_t - r_t)\,\text{DIS} \tag{88}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa x_t,\ \text{NKPC} \tag{89}$$

$$i_t \geq 0,\ \text{ZLB} \tag{90}$$

- Have ZLB without Taylor rule

First best with Taylor rule

- Can show that maximizing household welfare is approximately equivalent to minimizing following loss function (see Chapter 4 Appendix of Gali (2012))

$$L = \frac{1}{2}\int_0^\infty e^{-\delta t}\left(x_t^2 + \lambda \pi_t^2\right) dt \tag{91}$$

- If $r_t > 0$, can obtain first best $(x_t, \pi_t) = (0, 0)$ via Taylor rule

$$i_t = r + \phi_x x_t + \phi_x \pi_t,\ \phi_x > 1 \tag{92}$$

- there is no trade-off between the stabilization of inflation and the stabilization of the welfare-relevant output gap (the gap between actual output and efficient output) for central banks – **divine conincidence** – see Blanchard & Gali (2007)

Allow for a liquidity trap

- **Assume** economy starts off in a liquidity trap, but exits at time T [different from Benhabib, Schmitt-Grohe & Uribe (2002), where liquidity trap arises endogenously, but output is fixed]

$$r_t = \begin{cases} \underline{r} < 0, & t \in [0, T) \\ \overline{r} > 0, & t \geq T \end{cases} \tag{93}$$

What do mean by no commitment?

- Central bank cannot credibly announce plans about the future – e.g. cannot say is will raise interest rates if such and such happens
- Central bank optimizes as it goes as along

Key assumptions

- At time of exit from liquidity trap, attain first best, i.e. $\forall t \geq T$, $(x_t, \pi_t) = (0, 0)$
- this gives us a terminal boundary condition
- solve system of ode's for x_t and π_t for $t < T$:

$$\frac{dx_t}{dt} = -\sigma^{-1}(\underline{r} + \pi_t)\,\text{DIS} \tag{94}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa x_t,\ \text{NKPC} \tag{95}$$

Depression & Deflation I

Proposition 1

Consider a liquidity trap scenario, with $r_t < 0$ for $t < T$ and $r_t \geq 0$ for $t \geq T$. Let π_t^{nc} and x_t^{nc} denote the equilibrium outcome without commitment. Then inflation and output are zero after $t = T$ and strictly negative before that: $\pi_t^{nc} = x_t^{nc} = 0,\ t \geq T,\ \pi_t^{nc} < 0,\ x_t^{nc} < 0,\ t < T$. Moreover, π_t and x_t are strictly increasing in t for $t < T$. In the limit as $T \to \infty$, if the natural rate satisfies $\int_0^T r(t;T)ds \to -\infty$, then $\pi_0^{nc}, x_0^{nc} \to -\infty$

- The no commitment equilibrium features depression ($x_t^{nc} < 0,\ t < T$) and deflation ($\pi_t^{nc} < 0,\ t < T$). The longer the liquidity trap lasts ($T \to \infty$), the more severe the depression and deflation.
- The loss function diverges to ∞

Depression & Deflation II

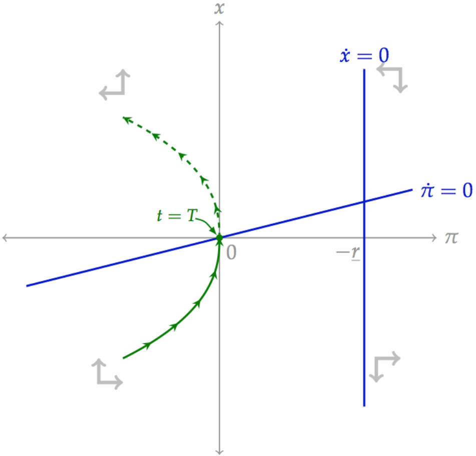


Figure 1: The equilibrium without commitment, featuring $i(t) = 0$ for $t \leq T$ and reaching $(0,0)$ at $t = T$.

Intuition

- real interest rate is set too high during the liquidity trap – depresses consumption
- effect of depressed consumption accumulates over time.
- deflation makes depression more severe by raising the real interest rate even more, further depressing output, leading to even more deflation, in a vicious cycle.

More on non-commitment

- Above outcome coincides with the optimal solution with commitment, if one constrains the problem by imposing $(\pi_T, x_T) = (0, 0)$. Therefore, we see that the ability to commit to outcomes within the interval $t \in [0, T)$ is irrelevant. Also, the ability to commit once $t = T$ is reached is also irrelevant.
- What is crucial is the ability to commit ex ante at $t < T$ to outcomes for $t = T$ – making commitments about policy after the liquidity trap during the trap would change the outcome

Value of commitment

- Look at a simple non-optimal policy with commitment

$$\forall t \geq 0,\ \pi_t = -\underline{r} > 0,\ x_t = -\frac{1}{\kappa}\underline{r} > 0 \tag{96}$$

- By design, output gap vanishes in flexible price limit ($\kappa \to \infty$)
- Nominal interest rate

$$i_t = r_t + \pi_t = \begin{cases} 0, & t < T \\ \overline{r} - \underline{r} > \overline{r} > 0, & t \geq T \end{cases} \tag{97}$$

- Nominal interest rate is zero during the liquidity trap and increased to keep inflation constant and positive after the trap
- Loss function given by

$$L = \frac{1}{2}\int_0^\infty e^{-\rho t}(x_t^2 + \lambda \pi_t^2)dt = \frac{1}{2}\int_0^\infty e^{-\rho t}(\underline{r}^2 + \lambda \frac{\underline{r}^2}{\kappa^2})dt \tag{98}$$

$$= \frac{\underline{r}^2}{2\delta}\left(1 + \frac{\lambda}{\kappa^2}\right) \to \frac{\underline{r}^2}{2\delta},\ as\ \kappa \to \infty \tag{99}$$

- An improvement on optimal policy with no commitment

Optimal Monetary Policy with Commitment

- How much better off can we be with an optimal policy under commitment
- Objective function

$$\inf_{(i_t)_{t\geq 0}} \int_0^\infty e^{-\rho t}(x_t^2 + \lambda \pi_t^2)dt \tag{100}$$

- State equations

$$\frac{dx_t}{dt} = \sigma^{-1}(i_t - r_t - \pi_t) \tag{101}$$

$$\frac{d\pi_t}{dt} = \rho \pi_t - \kappa x_t \tag{102}$$

$$i_t \geq 0 \tag{103}$$

- x_0 and π_0 are free, i.e. unconstrained

Deterministic Control Problem I

- Hamiltonian

$$\mathcal{H} = \frac{1}{2}(x_t^2 + \lambda\pi_t^2) + \mu_{x,t}\sigma^{-1}(i_t - r_t - \pi_t) + \mu_{\pi,t}(\rho\pi_t - \kappa x_t) - \mu_{i,t} i_t \quad (104)$$

- x_0 and π_0 are free, i.e. unconstrained $\Rightarrow$ associated Lagrange multipliers are zero (at date-0), i.e. $\mu_{x,0} = \mu_{\pi,0} = 0$

$$0 = \mathcal{H}_i \quad (105)$$

$$0 = \mu_{i,t} i_t, \text{ complementary slackness} \quad (106)$$

$$0 = \mathcal{H}_\pi + \frac{d\mu_\pi}{dt} - \rho\mu_\pi \quad (107)$$

$$0 = \mathcal{H}_x + \frac{d\mu_x}{dt} - \rho\mu_x \quad (108)$$

- Also need two transversality conditions (for $\mu_{x,t}$ and $\mu_{\pi,t}$)

Deterministic Control Problem II

- Simplify

$$\mu_{x,t}\sigma^{-1} = \mu_{i,t} \tag{109}$$

$$0 = \mu_{i,t} i_t \tag{110}$$

$$\frac{d\mu_{x,t}}{dt} = -x_t + \kappa\mu_{\pi,t} + \rho\mu_{x,t} \tag{111}$$

$$\frac{d\mu_{\pi,t}}{dt} = -\lambda\pi_t + \sigma^{-1}\mu_{x,t} \tag{112}$$

- Equations for state variables

$$\frac{dx_t}{dt} = \sigma^{-1}(i_t - \pi_t - r_t) \tag{113}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa x_t \tag{114}$$

Solution approach

1. Liquidity Trap, $t \in [0, T)$
2. ZLB holds, but LT has ended, $t \in [T, \hat{T})$
3. ZLB no longer holds, $t \in [\hat{T}, \infty)$

- Solve backwards in time and draw separate phase diagram for each time interval
- At the start time for each phase diagram there will be a unique (x, π), which satisfies transversality conditions
- Via choosing a time path for i_t, central bank can get onto a trajectory, which passes through (x, π)'s above

ZLB no longer holds, $t \in [\hat{T}, \infty)$ I

- $(x_{\hat{T}}, \pi_{\hat{T}})$ given by whatever happened before – take as fixed
- ZLB does not bind, i.e. $i_t > 0$, and so $\mu_{i,t} = 0$. Hence, $\mu_{x,t} = 0$
- Have following system of equations

$$0 = -x_t + \kappa \mu_{\pi,t} \tag{115}$$

$$\frac{d\mu_{\pi,t}}{dt} = -\lambda \pi_t \tag{116}$$

$$\frac{dx_t}{dt} = \sigma^{-1}(i_t - \pi_t - r_t) \tag{117}$$

$$\frac{d\pi_t}{dt} = \rho \pi_t - \kappa x_t \tag{118}$$

ZLB no longer holds, $t \in [\hat{T}, \infty)$ II

- Simplify

$$\kappa^{-1} \frac{dx_t}{dt} = -\lambda \pi_t \tag{119}$$

$$\frac{dx_t}{dt} = \sigma^{-1}(i_t - \pi_t - r_t) \tag{120}$$

$$\frac{d\pi_t}{dt} = \rho \pi_t - \kappa x_t \tag{121}$$

ZLB no longer holds, $t \in [\hat{T}, \infty)$ III

- Simplify further
  - interest rate rule [same interest rate condition as in Clarida, Gali and Gertler (1999)]

$$i_t = r_t + (1 - \kappa\sigma\lambda)\pi_t \tag{122}$$

  - linear system

$$\frac{dx_t}{dt} = -\kappa\lambda\pi_t \tag{123}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa x_t \tag{124}$$

- Solve linear system and get a stable saddle path

$$x_t = \phi\pi_t,\ \phi = \frac{\rho + \sqrt{\rho^2 + 4\lambda\kappa^2}}{2\kappa} > \frac{\rho}{\kappa} \tag{125}$$

ZLB no longer holds, $t \in [\hat{T}, \infty)$ IV

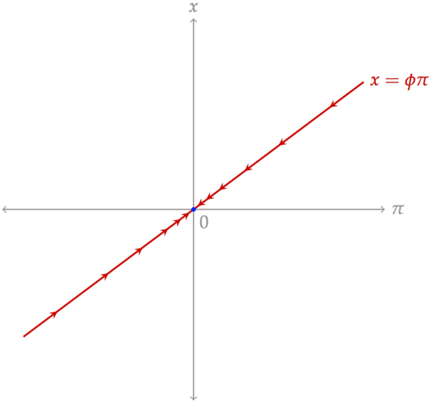


Figure 4: The solution without the ZLB constraint.

ZLB holds, but LT has ended, $t \in [T, \hat{T})$ I

- $i_t = 0$
- (x_T, π_T) given from previous outcomes
- linear system

$$\frac{dx_t}{dt} = -\sigma^{-1}(\bar{r} + \pi_t) \tag{126}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa x_t \tag{127}$$

- $x_T > \phi\pi_T$. The optimum policy attempts to reach the red line as quickly as possible, by setting the nominal interest rate to zero until $x_t = \phi\pi_t$

ZLB holds, but LT has ended, $t \in [T, \hat{T})$ II

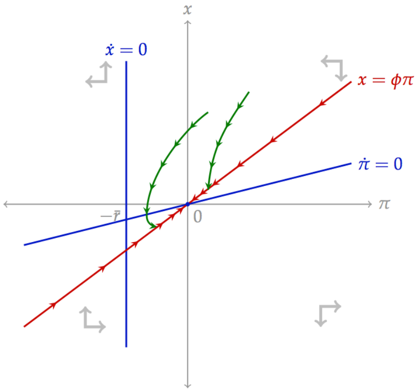


Figure 5: The solution for $t > T$ with the ZLB constraint.

Liquidity Trap, $t \in [0, T)$ I

- linear system

$$\frac{dx_t}{dt} = -\sigma^{-1}(\bar{r} + \pi_t) \tag{128}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa x_t \tag{129}$$

- (x_T, π_T) given and (x_0, π_0) free

Liquidity Trap, $t \in [0, T)$ II

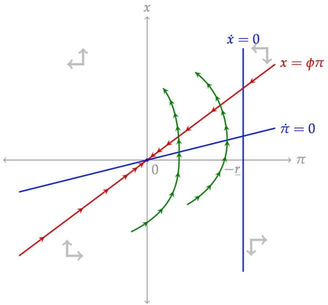


Figure 6: The solution for $t \leq T$ and $r(t) = -\bar{r} < 0$ with the ZLB constraint binding.

Summary

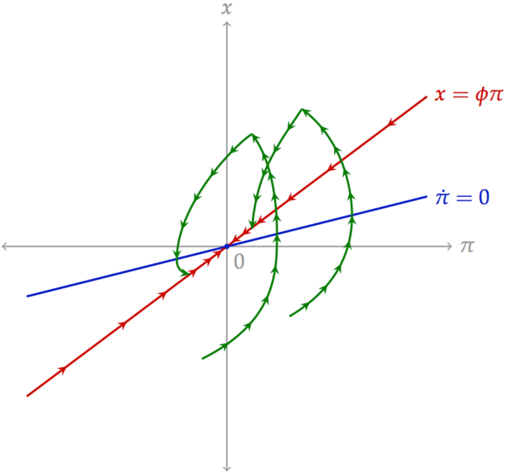


Figure 7: Two possible paths of the solution for $t \geq 0$.